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The system of Diophantine equations $(u - 1)x^2 - 4uy^2 = -12u - 8$ and $(u + 2)x^2 - 4uy^2 = -12u + 8$

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Abstract : Let $u \geq 5$ be an odd integer. The three numbers $u-2, u+2$ and $4u$ have the property that the product of any two distinct, increased by 4, is a perfect square. That property allows the solvability of the Diophantine quations $(u-2)x^2 - 4uy^2 = -12u - 8$ and $(u+2)x^2 - 4uz^2 = -12u - 8$. The integers solutions of the system of these two equations are given by $x = \pm 2$, $\pm(4u^2 - 2)$, $y = \pm 2$, $\pm(2u^2 - 2u - 2)$, $z = \pm 2$, $\pm(2u^2 + 2u - 2)$. We prove with the aid of simultaneous rational approximations and linear forms in logarithms of quadratic numbers that there is no other solution.

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1 Introduction

In paper [2], we have studied the system of Diophantine equations

$$3x^2 - 20y^2 = -68$$

and

$$7x^2 - 20z^2 = -52,$$

the discussion involving clearly the well-known following notion :

Definition 1 : Let w be a nonzero integer. A set of v positive integers $\{a_1, .., a_v\}$ is called a $D(w) - v - tuple$ if $a_i a_j + w$ is a square for all i and j with $1 \leq i < j \leq v$.

Looking at the coefficients of x^2, y^2 and z^2 in the equations above, we can write respectively

$$3 = u - 2, \qquad 7 = u + 2, \qquad 20 = 4u$$

with $u = 5$.

Consider for all integer $u \geq 0$, the three numbers $u - 2,$ $u + 2$ and $4u$.
If $u \in \{0, 1, 2\}$, then the set of those three numbers is not a $D(4) - triple$; but if $u \geq 3$ that set forms a $D(4) - triple$. Thus, for all integer $u \geq 3$, we shall employ the following notations: $T_u = \{u - 2, u + 2, 4u\};$
$Q_u = \{u - 2, u + 2, 4u, d\};$ where d is a positive integer.

Suppose that Q_u is a $D(4) - quadruple$ with $d > 4u$; then, there exist integers x, y, z such that:

$$(1.1) \quad x^2 = 4ud + 4, \qquad y^2 = (u - 2)d + 4, \qquad z^2 = (u + 2)d + 4 .$$

Eliminating d from (1.1), we obtain the simultaneous Diophantine equations

$$(E_1) \qquad (u - 2)x^2 - 4uy2 = -12u - 8$$

And

$$(E_2) \qquad (u + 2)x^2 - 4uz^2 = -12u + 8.$$

where $u \geq 3$ is a positive integer.

We denote by $(\mathcal{E})$ the system of (E_1) and (E_2).

The objective of this paper is the generalization of [2]. More precisely, it deals with the complete treatment of the solvability of $(\mathcal{E})$.

If $u = 3$, we valid $(\mathcal{E})$ because of the results of [5].

On the other hand, when u is even: $u = 2U, (E_1)$ and (E_2) become respectively

$$(U-1)x^2 - 4Uy^2 = -12U - 4$$

and

$$(U + 1)x^2 - 4Uz^2 = -12U + 4.$$

So, it is easy to see that the three numbers $U - 1, U + 1$ and $4U$ form a $D(1) - triple$. Therefore, we may assume that $u \geq 5$ is an integer with u odd.

Then, we remark immediately that $(\mathcal{E})$ possesses the obvious solutions

$$(x, y, z) = (\pm 2, \pm 2, \pm 2).$$

Definition 2: The obvious solutions above are called the trivial solutions of $(\mathcal{E})$.

Replacing the trivial solutions of $(\mathcal{E})$ in (1.1), we get $d = 0$.

Definition 3: The solution $d = 0$ above is called the trivial extension from T_u to Q_u.

The problem of finding the nontrivial solutions of $(\mathcal{E})$ involves in an essential way the determination of the extension $d \in Q_u$, $d \neq 0$, from T_u to Q_u; the key of that problem is the utilization of the following conjecture claimed in [4] :

Conjecture 4 : There does not exist a $D(4) - quintuple$. Moreover, if $\{a, b, c, d\}$ is a $D(4) - quadruple$ with $a < b < c < d$, then

$$d = a + b + c + \frac{abc + rst}{2},$$

where r, s, t are positive integers defined by :

$ab + 4 = r^2; \quad ac + 4 = s^2; \quad bc + 4 = t^2.$

Applying the second assertion of this result to Q_u we get $d = 4u(4u^2 - 1)$.

In Section 2, we give the family of nontrivial solutions of each separate equation of $(\mathcal{E})$ (Propositions 9 and 10) by the same arguments as in the following lemma proved in [4]:

Lemma 5: Let $\{a,b,c\}$ be a $D(4)-triple$ where $0 < a < b < c$, and let r,s,t be positive integers defined by

$ab + 4 = r^2, ac + 4 = s^2, bc + 4 = t^2.$

There exist positive integers i_0, j_0 and $x_0^{(i)}, y_0^{(i)}, z_1^{(j)}, z_1^{(j)}$, $i = 1,\ldots,i_0$, $j = 1,\ldots,j_0$ with the following properties:

$P_1 -$ $(x_0^{(i)}, y_0^{(i)})$ and $(x_1^{(j)}, z_1^{(j)})$ are respectively solutions of

$$(1.2) \qquad ax^2 - cy^2 = 4(a - c)$$

and

$$(1.3) \qquad bx^2 - cz^2 = 4(b - c).$$

$P_2 - x_0^{(i)}, y_0^{(i)}, x_1^{(j)}, z_1^{(j)}$, satisfy the following inequalities:

$$(1.4) \qquad 1 \le y_0^{(i)} \le \sqrt{\frac{a(c-a)}{s-2}}, |x_0^{(i)}| \le \sqrt{\frac{(s-2)(c-a)}{a}}.$$

$$(1.5) \qquad 1 \le z_1^{(j)} \le \sqrt{\frac{b(c-b)}{t-2}}, |x_1^{(j)}| \le \sqrt{\frac{(t-2)(c-b)}{b}}.$$

$P_3 -$ If (x,y) and (x,z) are positive solutions of (1.2) and (1.3) respectively, then there exist $i \in \{1,\ldots,i_0\}, j \in \{1,\ldots,j_0\}$ and integers $m,n \ge 0$ such that

$$(1.6) \qquad x\sqrt{a} + y\sqrt{c} = (x_0^{(i)}\sqrt{a} + y_0^{(i)}\sqrt{c})\left(\frac{s + \sqrt{ac}}{2}\right)^m.$$

$$(1.7) \qquad x\sqrt{b} + z\sqrt{c} = (x_1^{(j)}\sqrt{b} + z_1^{(j)}\sqrt{c})\left(\frac{t + \sqrt{bc}}{2}\right)^n.$$

Assuming the solvability of $(\mathcal{E})$, we introduce in Section 3 the recursive sequences connected to the families of nontrivial solutions of (E_1) and (E_2)

(Proposition 11). The solvability of $(\mathcal{E})$ in nontrivial integers imposes clearly that $x \equiv 0 \pmod 2$. Therefore, our study will be based on $X = \frac{x}{2}$. Thus, in Section 4 we put up a vision of linear forms in logarithms of quadratic numbers (Theorem 13). We study in Section 5 simultaneous rational approximations (Theorem 15) with the aid of the following result proved in [7] (see also [6]):

Proposition 6: If $u \geq 63$ is an integer, then the numbers

$$\theta_1 = \sqrt{\frac{u-2}{u}}$$

and

$$\theta_2 = \sqrt{\frac{u+2}{u}}$$

satisfy

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > (22.6u)^{-1} q^{-1-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = \frac{\log(11.2u)}{\log(0.197u^2)} < 1.$$

In Section 6, we describe the nontrivial solutions of $(\mathcal{E})$ (Theorem 18): for $u < 63$, we bound $\log X$ and also the positive integer n in terms of which X is expressed by (3.6) below (Lemma 16) ; so, we use the following results proved respectively in [1] and [3]:

Theorem 7: For a linear form $\Lambda \neq 0$ in logarithms of k algebraic numbers $\alpha_1, \dots, \alpha_k$ with rational coefficients $b_1, \dots, b_k$, we have:

$$\log|\Lambda| \geq -18(k+1)!\, k^{k+1} (32\delta)^{k+2} h'(\alpha_1) h'(\alpha_2) \dots h'(\alpha_k) \log(2k\delta) \log b$$

where

$$b = \max(|b_1|, \dots, |b_k|);\ \delta = [Q(\alpha_1, \dots, \alpha_k) : Q]$$

and

$$h'(\alpha)=\frac{1}{\delta}max(h(\alpha),|\log\alpha|,1)$$

with the standard logarithmic Weil height $h(\alpha)$ of α .

Lemma 8: Let M be a positive integer. Let $\frac{p}{q}$ the convergent of the continued fraction expansion of θ such that $q>6M$. Put

$$\epsilon=\|\mu q\|-M\|\theta q\|$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon>0$, then the inequality

$$0<n\theta-m+\mu<AB^{-n}$$

has no solution in the range

$$\frac{log\left(\frac{Aq}{\epsilon}\right)}{logB}\leq n<M.$$

Next, for $u\geq 63$, we prove that the set Q_u is a $D(4)-quadruple$ if and only if $d=4u(4u^2-1)$ (Lemma 17). The paper is ended in Section 7 with the complete set of integer solutions of (E) (Theorem 19).

2 The families of nontrivial solutions of (E_1) and (E_2)

In this section, we give: using the arguments of lemma 5, the family of nontrivial solutions of each separate equation of (E).

2.1 The family of (E_1)

It is clear that if (x,y) is a solution of (E_1), then so is $(-x,-y)$. Therefore, we may assume that (x,y) is positive. We prove the following proposition:

Proposition 9: Let $u\geq 5$ be an odd integer. Then, the nontrivial solutions in pairs of natural numbers (x,y) of (E_1) comprise the values of the sequences $(x_m,y_m)(m\geq 1)$ by setting:

$$(2.1)\qquad x_m\sqrt{u-2}+2y_m\sqrt{u}=\left(\pm2\sqrt{u-2}+4\sqrt{u}\right)\left(u-1+\sqrt{u(u-2)}\right)^m.$$

Proof. Let $(x,y)\in N^2$ be a solution of equation (E_1). Then, taking

$$a=u-2,\qquad c=4u,\qquad s=2(u-1)$$

in lemma 5, we see that there exists a particular solution (x_0, y_0) of (E_1) satisfying the following inequalities:

$$(2.2) \qquad 1 \leq y_0 \leq \sqrt{\frac{(u-2)(4u-u+2)}{2(u-1)-2}} = \sqrt{\frac{3u+2}{2}} .$$

$$(2.3) \qquad |x_0| \leq \sqrt{\frac{(2(u-1)-2)(4u-u+2)}{u-2}} = \sqrt{2(3u+2)} .$$

Then, by (1.6) we have

$$(2.4) \quad x\sqrt{u-2} + 2y\sqrt{u} = \left(x_0\sqrt{u-2} + 2y_0\sqrt{u}\right)\left(u-1+\sqrt{u(u-2)}\right)^m .$$

where $m \geq 1$ is an integer. But from (2.2), we have in particular

$$y_0 \leq \sqrt{\frac{3u+2}{2}} \Leftrightarrow -2y_0^2 \geq -3u-2.$$

In that last inequality, $u \geq 5$ imposes

$$-2y_0^2 \geq -3u-2 \geq -17.$$

This is only possible if $y_0 \leq \sqrt{\frac{17}{2}}$. Therefore there are only two values of y_0:

$y_0 = 1$: then, from (E_1) we have $x_0 = \pm 2\sqrt{\frac{-2(u+1)}{u-2}}$ which is not an integer;

$y_0 = 2$: here $x_0 = \pm 2$.

It follows that from (2.4) we have

$$(2.5) \quad x\sqrt{u-2} + 2y\sqrt{u} = \left(\pm 2\sqrt{u-2} + 4\sqrt{u}\right)\left(u-1+\sqrt{u(u-2)}\right)^m .$$

Taking $x = x_n,\ y = y_n$ in (2.5), we obtain (2.1). ∎

2.2 The family of (E_2)

It is also clear that if (x, z) is a solution of (E_2), then so is $(-x, z)$. Therefore, we may assume that (x, z) is positive. We prove also the following proposition:

Proposition 10: Let $u \geq 5$ be an odd integer. Then, the nontrivial solutions in pairs of natural numbers (x, z) of (E_2) comprise the values of the sequences $(x_n, z_n)(n \geq 1)$ by setting:

$$(2.6) \qquad x_n\sqrt{u+2} + 2z_n\sqrt{u} = \left(\pm 2\sqrt{u+2} + 4\sqrt{u}\right)\left(u+1 + \sqrt{u(u+2)}\right)^m.$$

Proof. Let $(x, z) \in N^2$ be a solution of equation (E_2). Then, taking

$b = u+2, \qquad c = 4u, \qquad t = 2(u+1)$

in lemma 5, we see that there exists a particular solution (x_1, z_1) of (E_2) satisfying the following inequalities:

$$(2.7) \qquad 1 \leq z_1 \leq \sqrt{\frac{(u+2)(4u-u-2)}{2(u+1)-2}} = \sqrt{\frac{(u+2)(3u-2)}{2u}}.$$

$$(2.8) \qquad |x_1| \leq \sqrt{\frac{(2(u+1)-2)(4u-u-2)}{u+2}} = \sqrt{\frac{2u(3u-2)}{u+2}}.$$

Then, by (1.7) we have

$$(2.9) \quad x\sqrt{u+2} + 2z\sqrt{u} = \left(x_1\sqrt{u+2} + 2z_1\sqrt{u}\right)\left(u+1 + \sqrt{u(u+2)}\right)^n.$$

where $n \geq 1$ is an integer. But from (2.7), doing as above in proof of proposition 9, we obtain $z_1 \leq \sqrt{\frac{91}{10}}$. Therefore z_1 has only three possible values:

$z_1 = 1$: then, from (E_2) we have $x_1 = \pm 2\sqrt{\frac{-2(u-1)}{u+2}}$ which is not an integer;

$z_1 = 2$: whence $x_1 = \pm 2$.

$z_1 = 3$: then $x_1 = \pm 2\sqrt{\frac{2(3u+1)}{u+2}}$ which is not a solution because of (2.8).

It follows that from (2.9) we have

$$(2.10) \quad x\sqrt{u+2} + 2z\sqrt{u} = \left(\pm 2\sqrt{u+2} + 4\sqrt{u}\right)\left(u+1 + \sqrt{u(u+2)}\right)^n.$$

Taking $x = x_n, z = z_n$ in (2.10), we obtain (2.6).

3 Recursive sequences connected to the general solutions of (E_1) and (E_2)

In this section, we consider the trivial solutions of (E) and formulae (2.1) and (2.6).

Proposition 11: Let $u \geq 5$ be an odd integer for which equations (E_1) and (E_2) have the general solutions given by definition 3 and formulae (2.1) and (2.6) respectively. Then, besides the trivial sequences $x_0 = \pm 2$, the sequences (x_m) and (x_n) verify respectively the following recursive formulae:

$i)\ x_1(m+2) = 2(u-1)\,x_1(m+1) - x_1 m$,

$ii)\ x_1(n+2) = 2(u+1)\,x_1(n+1) - x_1 n$,

for some integers $m, n \geq 1$.

Moreover, for these formulae, we have

$m \equiv n \equiv 0, 2 (mod\ 4).$

In other words,

$m = n = 4u \ or\ m = n = 4u + 2.$

Proof: It suffices to prove (i), the proof of (ii) is similar. We have of course $x_0 = \pm 2$, $x_1 = 6u - 2$ or $2u + 2$ and we see that, even if $x_0 = -2$, other values of x_n can be positive.

Next, relation (2.1) can be expressed in the form:

$$x_{m+1}\sqrt{u-2} + 2y_{m+1}\sqrt{u} = 2(\pm\sqrt{u-2} + 2\sqrt{u})\left(u - 1 + \sqrt{u(u-2)}\right)^{m+1}$$

or

$$x_{m+1}\sqrt{u-2} + 2y_{m+1}\sqrt{u} = (x_m\sqrt{u-2} + 2y_m\sqrt{u})\left(u - 1 + \sqrt{u(u-2)}\right)$$

whence

$$x_{m+2}\sqrt{u-2} + 2y_{m+2}\sqrt{u} = (x_m\sqrt{u-2} + 2y_m\sqrt{u})\left(u - 1 + \sqrt{u(u-2)}\right)^2.$$

But

$$(u-1)^2 + u^2 - 2u = (2u^2 - 4u + 2) - 1.$$

Therefore

$$x_{m+2}\sqrt{u-2}+2y_{m+2}\sqrt{u}=(x_m\sqrt{u-2}+2y_m\sqrt{u})\left[2(u-1)^2+2(u-1)\sqrt{u(u-2)}-1\right]$$

so that

$$x_{m+2}\sqrt{u-2}+2y_{m+2}\sqrt{u}=2(u-1)(x_{m+1}\sqrt{u-2}+2y_{m+1}\sqrt{u})-(x_m\sqrt{u-2}+2y_m\sqrt{u})$$

which gives:

$$x_{m+2} = 2(u-1)x_{m+1}-x_m;$$

$$y_{m+2} = 2(u-1)y_{m+1}-y_m.$$

This proves (i) and the first part of the proposition.

Considering relation (2.1), we must find m such that

$$(3.1) \qquad (u-2)x_m^2-4uy_m^2 = -12u-8.$$

Using (i), modulo $2(u-1)$ we have:

m	0	1	2	3	4
x_m	2	-4	-2	4	2

Here, we see that the sequence (x_m) is periodic with period 4. Then from (3.1) we obtain

$$(3.2) \qquad (u-2)x_m^2-4uy_m^2 = -20(mod\ 2(u-1)).$$

This implies

$$x_m \equiv \pm 2(mod\ 2(u-1))$$

which imposes $m \equiv 0,2(mod\ 4)$.

We consider now relation (2.6). As above, we find n such that

$$(3.3) \qquad (u+2)x_n^2-4uz_n^2 = -12u+8.$$

Using (ii), modulo $2(u+1)$ we have:

n	0	1	2	3	4
x_n	2	4	-2	4	2

,

and we see that, the sequence (x_n) is periodic with period 4. Then from (3.3) we obtain

$$(3.4) \qquad (u+2)x_n^2 - 4uz_n^2 = 20(mod\ 2(u+1)).$$

This implies

$x_n \equiv \pm 2(mod\ 2(u+1))$

which imposes $n \equiv 0, 2(mod\ 4)$;

therefore m and n are both even. In other words, we may write $m = n = 4u$ or $m = n = 4u+2$. This proves the second part of the proposition and completes the proof. ■

Remark 12 : Equations (E_1) and (E_2) impose that x is even; if we put $x = 2X$, then (E_1) and (E_2) become respectively

$$(F_1) \qquad (u-2)X^2 - uy^2 = -3u-2$$

and

$$(F_2) \qquad (u+2)X^2 - uz^2 = -3u+2.$$

To simplify our study, we consider from now on (F_1) and (F_2).

We denote by $(\mathcal{F})$ the system of equations (F_1) and (F_2). We shall employ:

$$(3.5) \quad X_m = \left(\frac{\epsilon\sqrt{u-2}+2\sqrt{u}}{2\sqrt{u-2}}\right)\left(u-1+\sqrt{u(u-2)}\right)^m - \left(\frac{-\epsilon\sqrt{u-2}+2\sqrt{u}}{2\sqrt{u-2}}\right)\left(u-1-\sqrt{u(u-2)}\right)^m.$$

$$(3.6) \quad X_n = \left(\frac{\epsilon\sqrt{u+2}+2\sqrt{u}}{2\sqrt{u+2}}\right)\left(u+1+\sqrt{u(u+2)}\right)^n - \left(\frac{-\epsilon\sqrt{u+2}+2\sqrt{u}}{2\sqrt{u+2}}\right)\left(u+1-\sqrt{u(u+2)}\right)^n.$$

where $m, n \geq 0$ are integers and $\epsilon \pm 1$.

We set

$X = X_m = X_n.$

Then, the values $X = \pm 1$ correspond to $m = n = 0$ in (3.5) and (3.6); those values are also called the *nontrivial solutions* of $(\mathcal{F})$.

In the ensuing of this paper, we seek the nontrivial values of X by methods using linear forms in logarithms of quadratic numbers and simultaneous rational approximations. According to the precedent proposition, we shall assume that m and n are both even ≥ 2.

4 Linear forms in logarithms of quadratic numbers

The present section is devoted to one important theorem of linear forms.

Theorem 13: Let $u \geq 5$ be an odd integer for which (F_1) and (F_2) have respectively nontrivial solutions given by (3.5) and (3.6). Let X be a nontrivial solution of (F) for some even integers $m, n \geq 2$. Then the linear form

$$(4.1) \qquad \Lambda = n\log\alpha_2 - m\log\alpha_1 + \log\alpha_3$$

with

$$(4.2) \qquad \alpha_1 = u - 1 + \sqrt{u(u-2)}, \quad \alpha_2 = u + 1 + \sqrt{u(u-2)}, \quad \alpha_3 = \frac{\left(\varepsilon\sqrt{u+2} + 2\sqrt{u}\right)\sqrt{u-2}}{\left(\varepsilon\sqrt{u-2} + 2\sqrt{u}\right)\sqrt{u+2}}$$

satisfies

$$(4.3) \qquad 0 < \Lambda < 0.5\alpha_2^{-2n}.$$

Moreover, the integer n verifies the inequality

$$(4.4) \qquad \frac{n}{2} < 10^4 \log n \log(2u-1)\log\left(77u^4\right).$$

Proof. Consider (3.5) and (3.6). Then (as $X = X_m = X_n$) we can write:

$$X = \left(\frac{\epsilon\sqrt{u+2} + 2\sqrt{u}}{\sqrt{u+2}}\right)\left(u + 1 + \sqrt{u(u+2)}\right)^n - \left(\frac{2\sqrt{u} - \epsilon\sqrt{u+2}}{\sqrt{u+2}}\right)\left(u + 1 - \sqrt{u(u+2)}\right)^{-n}$$

$$= \left(\frac{\epsilon\sqrt{u-2} + 2\sqrt{u}}{\sqrt{u-2}}\right)\left(u - 1 + \sqrt{u(u-2)}\right)^m - \left(\frac{2\sqrt{u} - \epsilon\sqrt{u-2}}{\sqrt{u-2}}\right)\left(u - 1 + \sqrt{u(u-2)}\right)^{-m}$$

If we put :

$$(4.5) \quad P = \left(\frac{\epsilon\sqrt{u+2} + 2\sqrt{u}}{\sqrt{u+2}}\right)\left(u + 1 + \sqrt{u(u+2)}\right)^n, \qquad Q = \left(\frac{\epsilon\sqrt{u-2} + 2\sqrt{u}}{\sqrt{u-2}}\right)\left(u - 1 + \sqrt{u(u-2)}\right)^m$$

then, that last relations give

(4.6) $P+\frac{3u-2}{4u+8}P^{-1} = Q+\frac{3u+2}{4u}Q^{-1}.$

Since

$$P-Q=\frac{3u+2}{4u}Q^{-1}-\frac{3u-2}{4u+8}P^{-1}.$$

$u \geq 5$ implies

$$P-Q>\frac{13}{28}Q^{-1}-\frac{13}{28}P^{-1}=\frac{13}{28}(P-Q)P^{-1}Q^{-1}.$$

and plainly $P > 1, Q > 1$ we must have $P > Q$. As we may assume that $n \geq 2$, the inequality $u \geq 5$ imposes

$$P \geq \frac{\sqrt{7}+2\sqrt{5}}{\sqrt{7}}(6+\sqrt{35})^2 > 605.$$

Relation (4.6) implies

$$Q > P-\frac{17}{28}Q^{-1} > P-\frac{17}{28}.$$

Hence

$$P-Q=\frac{3u+2}{4u}Q^{-1}-\frac{3u-2}{4u+8}P^{-1}<\frac{17}{28}\left(P-\frac{17}{28}\right)^{-1}-\frac{13}{28}P^{-1}<\frac{1}{2}P^{-1}.$$

It follows from

$$0<\frac{P-Q}{P}<\frac{1}{2}P^{-2}=\frac{1}{2}\times 605^{-2}$$

that

$$0<\log\frac{P}{Q}=-\log\left(1-\frac{P-Q}{Q}\right)<\frac{1}{2}P^{-2}+\left(\frac{1}{2}P^{-2}\right)^2<\frac{1}{2}P^{-2}\left(1+\frac{1}{2}\times 605^{-2}\right)<0.5P^{-2}.$$

Since

$$P^{-2}<\left(6+\sqrt{35}\right)^{-2n}.$$

substituting from (4.5) we obtain (4.1).

It remains to show inequality (4.4). Here, we have to apply theorem 7 with $k = 3$ and relations (4.2). Thus, we can take $\delta = 4, b = n$ and

$h'(a_1) = \frac{1}{2}\log a_1, h'(a_2) = \frac{1}{2}\log a_2.$

Denoting the conjugate of a_2 by a_2^{σ}, we may write

$$h'(a_3) \le \frac{1}{4}[\log((3u+2)^2(u+2)^2) + \log[(a]_3 a_3^{\sigma})] < \frac{1}{4}\log\left(16u^2(3u+2)(u+2)\right) < \frac{1}{4}\log(77u^4).$$

Then, by theorem 7 we have

$$\log\Lambda > -18 \times 4! \frac{\times 3^4 (32 \times 4)^{5.1}}{2} \log a_1 \frac{1}{2}\log a_2 . \frac{1}{4}\log(77u^4)$$

$log(24)\log n.$

But

$a_1 = u - 1 + \sqrt{u^2 - 2u} > 2u - 1.$

Therefore, from (4.1) we deduce that

$$\frac{n}{\log n} < 1 \times 2 \times 10^{14} log(2u-1)\log(77u^4). \blacksquare$$

Lemma 14: With the notations of the preceding theorem, we have:

$(4.7) \qquad 0 < \Lambda \Rightarrow n \ge m.$

Proof. Suppose that $n < m$; then we have:

$$\Lambda = n\log a_2 - m\log a_1 + \log a_3 < m\log a_2 - m\log a_1 + \log a_3$$
$$= -m(\log a_1 - \log a_2) + \log a_3 < \log a_1 - \log a_2 + \log a_3$$
$$= \log a_2 + \log(a_1 a_3) < 0,$$

which contradicts inequality (4.7). $\blacksquare$

5. Simultaneous rational approximations

It is clear that if (X, y, z) is a nontrivial positive solution of (F), so is $(-X, -y, -z)$. Thus, we can suppose that (X, y, z) is positive.

Let (X, y, z) be a nontrivial positive solution of (F). Then, we may write (F_1) and (F_2) respectively in the form

$$\sqrt{\frac{u-2}{u}} - \frac{y}{X} = \left(\frac{u-2}{u} - \frac{y^2}{X^2}\right)\left(\sqrt{\frac{u-2}{u}} + \frac{y}{X}\right)^{-1}$$

and

$$\sqrt{\frac{u+2}{u}}-\frac{z}{X}=\left(\frac{u+2}{u}-\frac{z^2}{X^2}\right)\left(\sqrt{\frac{u+2}{u}}+\frac{z}{X}\right)^{-1}$$

Then, taking the absolute values in these two last equalities, we obtain respectively

$$\text{(5.1)}\qquad \left|\sqrt{\frac{u-2}{u}}-\frac{y}{X}\right|=\frac{1}{uX^2}|-3u-2|\left|\sqrt{\frac{u-2}{u}}+\frac{y}{X}\right|^{-1}$$

and

$$\text{(5.2)}\qquad \left|\sqrt{\frac{u+2}{u}}-\frac{z}{X}\right|=\frac{1}{uX^2}|-3u+2|\left|\sqrt{\frac{u+2}{u}}+\frac{z}{X}\right|^{-1}.$$

Next, we prove the following theorem :

Theorem 15: Let $u \geq 63$ be an odd integer such that (X,y,z) is a nontrivial positive solution of $(\mathcal{F})$. Then (X,y,z) satisfies

$$\text{(5.3)}\quad (22.6u)^{-1}X^{-1-\lambda}2^{\lambda} < \max\left\{\left|\sqrt{\frac{u-2}{u}}-\frac{2y}{x}\right|,\left|\sqrt{\frac{u+2}{u}}-\frac{2z}{x}\right|\right\} < 1.5X^{-2}$$

with

$$\lambda = \frac{\log(11.2u)}{\log(0.197u^2)} < 1.$$

Proof. Taking $p_1 = y, p_2 = z$ and $q = X$ in (5.1) and (5.2), by proposition 6. we see that the solution (X,y,z) of $(\mathcal{F})$ satisfies

$$\max\left\{\left|\sqrt{\frac{u-2}{u}}-\frac{y}{X}\right|,\left|\sqrt{\frac{u+2}{u}}-\frac{z}{X}\right|\right\} > (22.6u)^{-1}X^{-1-\lambda}$$

with

$$\lambda = \frac{\log(11.2u)}{\log(0.197u^2)}.$$

It is also clear that $u \geq 63$ implies $\lambda < 1$. This proves the first inequality of (5.3).

Let us show now the last inequality of (5.3). From (5.2) we have

$$(5.4) \qquad \left|\sqrt{\frac{u+2}{u}}-\frac{z}{X}\right| = \frac{1}{uX^2}|-3u+2|\frac{1}{\left|\sqrt{\frac{u+2}{u}}+\frac{z}{X}\right|} \leq \frac{1}{uX^2}(|-3u|+2)\frac{1}{2\sqrt{1+\frac{2}{u}}} < 1.55X^{-2}$$

(since $u \geq 63, \sqrt{\frac{u+2}{u}} > 1$ and $\frac{z}{X} > 0$).

Doing as above with equality (5.1), we also obtain

$$(5.5) \qquad \left|\sqrt{\frac{u-2}{u}}-\frac{y}{X}\right| < 1.55X^{-2}$$

From (5.4) and (5.5), we see that

$$\max\left\{\left|\sqrt{\frac{u-2}{u}}-\frac{y}{X}\right|,\left|\sqrt{\frac{u+2}{u}}-\frac{z}{X}\right|\right\} < 1.5\, X^{-2}$$

so that (5.3) holds and the proof is now complete.■

6 The nontrivial solutions of $(\mathcal{F})$

In this section, we have to examine two cases: $5 \leq u < 63$ and $u \geq 63$.

6.1 The case $5 \leq u < 63$

In this case, we go on to prove the following lemma:

Lemma 16: Let u be an odd integer such that $5 \leq u < 63$. With the notations and hypotheses of theorem 13, if $n \geq 2$ is an even integer satisfying (4.3) and (4.4), then $n = 2$.

Proof. Suppose that $n \neq 2$, that is $n \geq 4$. Then, inequalities (4.3) imply, after dividing by $\log a_1$ that

$$0 < n\theta - m + \mu < AB^{-n}.$$

With

$\theta = \frac{\log a_2}{\log a_1}, \mu = \frac{\log a_3}{\log a_1}, A = \frac{0.5}{\log a_1}, B = a_2^2.$ But lemma 14 and the last equalities of proposition 11 imply, since $u \geq 5$,

$$n \geq m \geq 4u + 2 \geq 22.$$

As $u < 63$, it follows from (4.4) that $n < 5\times 10^{17}$. Thus, taking $M = 5\times 10^{17}$ in lemma 8, we see that we have to examine 29 cases for which the second convergent of θ with $q > 6M$ is needed only in two cases: $u = 5$ and $u \geq 7$, therefore $u \geq 5$. This implies $n < 14$ in which case the second step of reduction of lemma 8 with $M = 13$ imposes $n < 4$ which contradicts the supposition that $n \geq 4$. ■

6.2 The case $u \geq 63$

In this case, we prove also the following lemma :

Lemma 17: Let $u \geq 63$ be an odd integer. With the same notations and hypotheses as in lemma 16, the set Q_u is a $D(4) - quadruple$ if and only if $d = 4u(u^2 - 1)$.

Proof. If $d = 4u(u^2 - 1)$, then by definition 1, Q_u is a $D(4) - quadruple$.

Conversely, suppose that $d \neq 4u(u^2 - 1)$. Since X is a nontrivial solution of $(\mathcal{F})$, $-X$ is also a nontrivial solution of $(\mathcal{F})$. Therefore, we may suppose that X is positive. Then, from the first relation of (1.1) we have

$$4X^2 \neq 16u^2(u^2-1) + 4.$$

whence $(X > 0)$

$$X \neq 2u^2 - 1.$$

Then, from (3.6) we may write (as $X = X_m = X_n$ is positive) $X = Y_n$ for $n \geq 2$, where

$$Y_n = \left(\frac{\sqrt{u-2} + 2\sqrt{u}}{2\sqrt{u-2}}\right)\left(u - 1 + \sqrt{u^2 - 2u}\right)^n - \left(\frac{-\sqrt{u-2} + 2\sqrt{u}}{2\sqrt{u-2}}\right)\left(u - 1 - \sqrt{u^2 - 2u}\right)^n$$

Therefore we have

$$Y_n > \left(u - 1 + \sqrt{u^2 - 2u}\right)^n - \left(u - 1 - \sqrt{u^2 - 2u}\right)^n > (2u - 3)^n.$$

Then, taking the logarithms, we see that

$$\log X > n \log(2u - 3).$$

But from proposition 11 we have in particular $n = 4u + 2$ so that

$$(6.5) \quad \log X > (4u + 2)\log(2u - 3).$$

Next, from theorem 15, we have the inequality

$$(6.6) \quad (22.6u)^{-1}X^{-1-\lambda} < 1.55X^{-2}, \qquad \lambda < 1$$

so that

$$X^{1-\lambda} < 35.03u,$$

and taking again the logarithms of this last inequality we see that

$$(6.7) \log X < \frac{\log(35.03u)}{1-\lambda}.$$

Since

$$1 - \lambda = 1 - \frac{\log(11.2u)}{\log(0.197u^2)} = \frac{\log(0.0175u)}{\log(0.197u^2)},$$

we have

$$\frac{1}{1-\lambda} = \frac{\log(0.197u^2)}{\log(0.0175u)} < \frac{2\log(0.444u)}{\log(0.0175u)}.$$

Thus, relations (6.5) and (6.7) imply

$$2u + 1 < \frac{\log(0.444u)\log(35.03u)}{\log(2u-3)\log(0.0175u)}.$$

Set

$$(6.8) \qquad \beta(u) = \frac{\log(0.444u)\log(35.03u)}{\log(2u-3)\log(0.0175u)}.$$

Then, from (6.8) we see that

$$2u - 3 < 35.03u, \qquad 0.0175u < 0.444u$$

so that $\beta(u)$ is decreasing. Further the inequality

$$\beta(u) = \beta(63) < 55$$

imposes $u < 27$ which contradicts the supposition that $u \geq 63$. ■

6.3 Description of nontrivial solutions of (F)

Theorem 18: Let $u \geq 5$ be an odd integer for which the Diophantine equations (F_1) and (F_2) have nontrivial solutions given respectively by (3.5) and (3.6). Then, all the nontrivial integer solutions of $(\mathcal{F})$ are given by:

$$\begin{cases} X = \pm(2u^2-1) \\ y = \pm(2u^2-2u-2) \\ z = \pm(2u^2+2u-2) \end{cases}.$$

Proof. Easy calculations show that formulae above give nontrivial solutions for $(\mathcal{F})$. Conversely, let X, y, z be nontrivial integers such that we have $(\mathcal{F})$. Then, with conjecture 4, we have got $d = 4u(4u^2-1)$ which yields the nontrivial solutions of $(\mathcal{F})$. Thus, from relations (1.1) we get:

$$\begin{cases} X^2 = 4u^4 - 4u^2 + 1 = (2u^2-1)^2 \\ y^2 = (2u^2-2u-2)^2 \\ z^2 = (2u^2+2u-2)^2 \end{cases}.$$

so that

$$\begin{cases} X = \pm(2u^2-1) \\ y = \pm(2u^2-2u-2) \\ z = \pm(2u^2+2u-2) \end{cases}$$

and lemmas 16 and 17 show that there is no other solution. ■

7 Complete set of solutions of $(\mathcal{E})$

Theorem 19: Let $u \geq 5$ be an odd integer. Then, all the integer solutions of $(\mathcal{E})$ are given by :

$$\begin{cases} x = \pm 2, \pm(4u^2-2) \\ y = \pm 2, \pm(2u^2-2u-2) \\ z = \pm 2, \pm(2u^2+2u-2) \end{cases}.$$

Proof. The trivial solutions of $(\mathcal{E})$ result from definitions 2 and 3 and the nontrivial solutions result (as $x = 2X$) from theorem 18.

Remark 20: If $u = 5$, we have studied in [2] the system $(\mathcal{E}_5)$ of equations $3x^2 - 20y^2 = -68$ and $7x^2 - 20z^2 = -52$. We have proved that all the solutions of

(E_5) are given by:

$$\begin{cases} x = \pm 2, \pm 9\text{[illegible]} \\ y = \pm 2, \pm 3\text{[illegible]} \\ z = \pm 2, \pm 5\text{[illegible]} \end{cases}.$$

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